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MEMORANDUM

A STUDY OF THE POSSIBLE USE OF THRUST VECTOR ROTATION

FOR "CUTOFF" FOR BALLISTIC MISSILES

By Aaron S. Boksenbom and Betty Jo Moore

Lewis Research Center
Cleveland, Ohio

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A STUDY OF THE POSSIBLE USE OF THRUST VECTOR ROTATION FOR "CUTOFF" FOR BALLISTIC MISSILES*

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SUMMARY

An alternate method to thrust termination for ballistic missiles is presented whereby the thrust vector is rotated to the insensitive direction. The miss distance is then fixed, independent of the magnitude of thrust. The chief advantages of this method, in addition to alleviating the requirement on elaborate thrust cutoff devices required for some propulsion systems, are that "cutoff" is reversible and that thrust vector rotation near the insensitive direction would give fine vernier control. For moderate missile rotational speeds, anticipation would be required resulting in some loss in maximum range. The effects of thrust vector rotation on the miss distances are expressed by relatively simple equations in vector form. The requirements on missile rotational speeds and the maximum loss in maximum range for various attainable rotational speeds are presented.

INTRODUCTION

In the launch phase of ballistic missiles fine trajectory control is achieved by appropriate termination of powered flight. In current practice the thrust is cut off when conditions are such that the subsequent unpowered trajectory intersects the target. An alternate method is presented in this report whereby the thrust vector is rotated to the insensitive direction. The miss distance would then be fixed, independent of the magnitude of thrust and of when or how burnout occurs.

In addition to alleviating the requirement for elaborate thrust-termination devices required of some propulsion systems, the chief advantages of this method are that "cutoff" is reversible and that thrust vector rotation around the insensitive direction can be used for vernier control. The chief problem in the use of this method is that either very high rotational speeds of the thrust vector are required or more anticipatory action must be used. There would then be some loss in the maximum range of the missile.

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To fix the miss distance along some line at the target, the insensitive direction of thrust is shown to be normal to a certain defined vector, which is in the direction of maximum sensitivity. To fix the miss distance on the two-dimensional surface of the earth, the thrust vector must be normal to a pair of vectors. These vectors are characteristics of the unpowered flight trajectory and can be calculated if the acceleration along the unpowered flight trajectory is a known function of the vector velocity, vector position, and time. The general equations derived in this report are thus valid unless unpredictable forces, such as those induced by random winds or by the missile attitude, are large.

SYMBOLS

a	acceleration
b	slope of straight line in E_1,E_2 plane
E	miss distance
$\nabla_v E$	gradient of E in velocity space, at fixed position and time
$\nabla_r E$	gradient of E in position space, at fixed velocity and time
$\partial E/\partial t$	derivative of E with respect to time, at fixed velocity and position
F	thrust
m	mass
$\hat{n}_1$	normal to $\hat{v}$, in $\hat{r},\hat{v}$ plane, in direction of increasing γ
$\hat{n}_2$	completion of right-hand trial $\hat{v},\hat{n}_1,\hat{n}_2$
r	position
t	time
v	velocity
$(\dot{\ })$	time derivative
α	azimuth direction measured counterclockwise from south
β	angle between $\overline{F}$ and $\overline{\nabla_v E}$, angle between $\overline{F}$ and $\hat{e}_1$

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γ	angle between $\hat{v}$ and horizontal plane
θ	colatitude
τ	time constant for thrust cutoff exponential decay
φ	longitude
$\hat{\varphi}$	unit vector horizontal plane, in direction of increasing φ

Subscripts:

min	minimum
0	initial condition

Vector notations:

$(\bar{\ })$	vector
$(\hat{\ })$	unit vector
$(\bar{\ }) \cdot (\bar{\ })$	vector dot product
$(\bar{\ }) \times (\bar{\ })$	vector cross product

ANALYSIS

General Equation for Miss Distance

The general features of the trajectories are shown schematically in figure 1. The wavy line is the powered flight trajectory. The solid lines are unpowered flight trajectories for burnout at various points along the powered flight trajectory. If, for example, burnout occurs at point A, the trajectory proceeds from A to impact at point B. There are several parameters at impact that might be of interest, such as miss distance, impact velocity, and so on. Figure 1 schematically shows the miss distance E_B. If the acceleration at any point along the unpowered trajectory is a known function of the vector velocity, vector position, and time at that point, then the impact parameters, such as E_B, can be considered as functions of the vector velocity, vector position, and time at point A, and E_B can be written as

$$E_B = E(\bar{v}_A, \bar{r}_A, t_A)$$

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If burnout occurs a short time later at point C, the trajectory proceeds from C to impact at point D; the miss distance is E_D and is given by

$$E_D = E(\bar{v}_C, \bar{r}_C, t_C)$$

Then,

$$E_D - E_B = E(\bar{v}_C, \bar{r}_C, t_C) - E(\bar{v}_A, \bar{r}_A, t_A)$$

As conditions at point C are close to those at point A, the preceeding equation can be expanded in terms of $\bar{v}_C - \bar{v}_A$, $\bar{r}_C - \bar{r}_A$, and $t_C - t_A$. In a compact vector notation, the first-order terms are

$$E_D - E_B = \overline{\nabla_v E}(t_A) \cdot (\bar{v}_C - \bar{v}_A) + \overline{\nabla_r E}(t_A) \cdot (\bar{r}_C - \bar{r}_A) + \frac{\partial E(t_A)}{\partial t} \cdot (t_C - t_A)$$

The argument t_A is used to indicate that the derivatives are evaluated at $\bar{v}_A$, $\bar{r}_A$, and t_A. The $\overline{\nabla_v E}$ and $\overline{\nabla_r E}$ vectors are such that the preceeding expansion would agree with the seven terms obtained in the usual expansion of E in terms of the three velocity coordinates, three position coordinates, and time. The effects of a velocity change and the effects of a position change are separated by the use of $\overline{\nabla_v E}$ and $\overline{\nabla_r E}$, respectively.

Several special cases can be derived from the previous equation. In figure 1, point C could be any arbitrary condition near point A. Then the preceeding equation can be written as

$$E - E_B = \overline{\nabla_v E}(t_A) \cdot (\bar{v} - \bar{v}_A) + \overline{\nabla_r E}(t_A) \cdot (\bar{r} - \bar{r}_A) + \frac{\partial E(t_A)}{\partial t} (t - t_A)$$

This equation is the general linearized guidance equation expanded around some expected burnout point $\bar{v}_A$, $\bar{r}_A$, t_A where $E_B = 0$. Burnout should then occur when $E = E_B = 0$ in the preceeding equation.

If points A and C are along an actual flight trajectory, then in the limit as $C \rightarrow A$,

$$\frac{\bar{v}_C - \bar{v}_A}{t_C - t_A} = \bar{a}_A$$

$$\frac{\bar{r}_C - \bar{r}_A}{t_C - t_A} = \bar{v}_A$$

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and

$$\frac{E_D - E_B}{t_C - t_A} = \left[\frac{dE(t_A)}{dt}\right]_{\text{along trajectory at } t_A}$$

Then, the previous equations give

$$\left[\frac{d}{dt}\, E(t_A)\right]_{\substack{\text{along trajectory} \\ \text{at } t_A}} = \overline{\nabla_v E}(t_A) \cdot \overline{a}_A + \overline{\nabla_r E}(t_A) \cdot \overline{v}_A + \frac{\partial}{\partial t}\, E(t_A) \tag{1}$$

If the trajectory is that of unpowered flight,

$$\left[\frac{d}{dt}\, E(t_A)\right]_{\substack{\text{along unpowered} \\ \text{trajectory at } t_A}} = \overline{\nabla_v E}(t_A) \cdot \overline{a}_{A,\text{unpowered}} + \overline{\nabla_r E}(t_A) \cdot \overline{v}_A + \frac{\partial E}{\partial t}(t_A)$$

$$= 0 \tag{2}$$

Along the unpowered flight trajectory $dE(t_A)/dt = 0$; that is, the path is along A to B in figure 1, which has a unique miss distance E_B. Subtracting equation (2) from equation (1) gives

$$\left[\frac{dE(t_A)}{dt}\right]_{\substack{\text{along trajectory} \\ \text{at } t_A}} = \overline{\nabla_v E}(t_A) \cdot (\overline{a}_A - \overline{a}_{A,\text{unpowered}})$$

$$= \overline{\nabla_v E}(t_A) \cdot \frac{\overline{F}(t_A)}{m(t_A)} \tag{3}$$

where $\overline{F}$ is thrust and m is mass.

Changing the variables t_A, $\overline{v}_A$, and $\overline{r}_A$ to the general variables t, $\overline{v}(t)$, and $\overline{r}(t)$ yields the final equation

$$\left[\frac{dE(t)}{dt}\right]_{\text{along trajectory}} = \overline{\nabla_v E}(t) \cdot \frac{\overline{F}(t)}{m(t)} \tag{4}$$

[redacted]

The miss distance $E(t) = E[\overline{v}(t), \overline{r}(t), t]$ and is defined for all times along the powered and unpowered flight trajectories. It is the miss distance that would be obtained if there were no thrust for all times greater than t.

Equation (4) is the basis for this report. It is evident that if the direction of the thrust vector were normal to the vector $\overline{\nabla_v E}$, then $dE/dt = 0$. That is, the miss distance would be fixed, independent of the magnitude of the thrust and of when or how burnout occurs.

It was shown that equation (4) can be written if the acceleration during unpowered flight is a known function of $\overline{v}$, $\overline{r}$, and t. For certain guidance functions and to obtain the general requirements on thrust cutoff or thrust vector rotation, only approximate values of dE/dt, and thus $\overline{\nabla_v E}$, are required. If the trajectory analysis omitted small forces, the effect on the unpowered flight trajectory would be small and the effect on the calculation of $\overline{\nabla_v E}$ would be small. The general form of equation (4) would not hold if unpredictable forces, such as those induced by random winds or by the missile attitude, were large.

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Two-Dimensional and Radial Miss Distances

For the actual two-dimensional miss distances on the rotating earth, the errors in a coordinate system relative to the target fixed on the earth are considered (fig. 2(a)). The two errors, E_1 and E_2, could be those in latitude and longitude or those down-range and cross-range. For each direction there is an equation similar to equation (4). Thus,

$$\left.\begin{aligned} \frac{dE_1(t)}{dt} &= \overline{\nabla_v E_1}(t) \cdot \left[\frac{\overline{F}(t)}{m(t)}\right] \\ \frac{dE_2(t)}{dt} &= \overline{\nabla_v E_2}(t) \cdot \left[\frac{\overline{F}(t)}{m(t)}\right] \end{aligned}\right\} \tag{5}$$

To fix the impact point on the rotating earth requires a unique thrust direction normal to the plane of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$.

If two unit orthogonal vectors $\hat{e}_1$ and $\hat{e}_2$ are placed in the plane of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$ shown in figure 2(b),

$$\left.\begin{aligned} \overline{\nabla_v E_1} &= (\overline{\nabla_v E_1} \cdot \hat{e}_1)\hat{e}_1 + (\overline{\nabla_v E_1} \cdot \hat{e}_2)\hat{e}_2 \\ \overline{\nabla_v E_2} &= (\overline{\nabla_v E_2} \cdot \hat{e}_1)\hat{e}_1 + (\overline{\nabla_v E_2} \cdot \hat{e}_2)\hat{e}_2 \end{aligned}\right\} \tag{6}$$

[redacted]

Equations (5) then become

$$\left.\begin{aligned}\frac{dE_1}{dt} &= (\overline{\nabla_v E_1} \cdot \hat{e}_1)\left(\hat{e}_1 \cdot \frac{\overline{F}}{m}\right) + (\overline{\nabla_v E_1} \cdot \hat{e}_2)\left(\hat{e}_2 \cdot \frac{\overline{F}}{m}\right) \\ \frac{dE_2}{dt} &= (\overline{\nabla_v E_2} \cdot \hat{e}_1)\left(\hat{e}_1 \cdot \frac{\overline{F}}{m}\right) + (\overline{\nabla_v E_2} \cdot \hat{e}_2)\left(\hat{e}_2 \cdot \frac{\overline{F}}{m}\right)\end{aligned}\right\} \tag{7}$$

If the thrust $\overline{F}(t)$ lies in a plane normal to $\hat{e}_2$, then $\hat{e}_2 \cdot \overline{F}(t) = 0$ and equations (7) become

$$\left.\begin{aligned}\frac{dE_1}{dt} &= (\overline{\nabla_v E_1} \cdot \hat{e}_1)\left(\hat{e}_1 \cdot \frac{\overline{F}}{m}\right) \\ \frac{dE_2}{dt} &= (\overline{\nabla_v E_2} \cdot \hat{e}_1)\left(\hat{e}_1 \cdot \frac{\overline{F}}{m}\right)\end{aligned}\right\} \tag{8}$$

The ratio of dE_2 to dE_1 is then

$$\frac{dE_2}{dE_1} = \frac{\overline{\nabla_v E_2} \cdot \hat{e}_1}{\overline{\nabla_v E_1} \cdot \hat{e}_1} \tag{9}$$

During the small time interval of "cutoff", for which $\nabla_v E_1$ and $\nabla_v E_2$ are relatively constant, the path in the E_1,E_2 plane is a straight line whose slope b is given by

$$b = \frac{\overline{\nabla_v E_2} \cdot \hat{e}_1}{\overline{\nabla_v E_1} \cdot \hat{e}_1} \tag{10}$$

Thus, if during thrust vector rotation the thrust vector is confined to a plane that is normal to the plane of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$, the possible impact points trace a straight-line path in the E_1,E_2 plane. For small errors this would be approximately a straight-line path on the earth. The slope of this path depends on the orientation of the plane of the thrust vector.

For the radial miss distance, reference is made to figure 2(a) in which $E^2 = E_1^2 + E_2^2$, by definition. For small errors E is close to the radial miss distance. The general equation of a straight line in the E_1,E_2 plane in terms of the slope b and the closest approach to the origin E_{min} (fig. 2(c)) is

$$E_2(t) = bE_1(t) \pm \sqrt{1 + b^2}\, E_{min} \tag{11}$$

The general equation for dE/dt may be written as

$$\left.\begin{aligned} \frac{dE}{dt} &= \left[\frac{E_1}{E}(\overline{\nabla_v E_1} \cdot \hat{e}_1) + \frac{E_2}{E}(\overline{\nabla_v E_2} \cdot \hat{e}_1)\right]\left(\hat{e}_1 \cdot \frac{\overline{F}}{m}\right) \\ &\text{where} \\ E^2 &= E_1^2 + \left(bE_1 \pm \sqrt{1 + b^2}\, E_{min}\right)^2 \\ &\text{or} \\ E^2 &= E_2^2 + \left(\frac{E_2 \mp \sqrt{1 + b^2}\, E_{min}}{b}\right)^2 \end{aligned}\right\} \tag{12}$$

For the case where $E_{min} = 0$, $E_2 = bE_1$ and

$$\left.\begin{aligned} \frac{dE}{dt} &= \pm\left[(\overline{\nabla_v E_1} \cdot \hat{e}_1)^2 + (\overline{\nabla_v E_2} \cdot \hat{e}_1)^2\right]^{1/2}\left(\frac{\overline{F}}{m} \cdot \hat{e}_1\right) \\ &\text{where } \pm \text{ is} \\ (\text{sign } E_1)&(\text{sign } \overline{\nabla_v E_1} \cdot \hat{e}_1) = (\text{sign } E_2)(\text{sign } \overline{\nabla_v E_2} \cdot \hat{e}_1) \end{aligned}\right\} \tag{13}$$

Since $\overline{\nabla_v E_1}$, $\overline{\nabla_v E_2}$, and $\hat{e}_1$ are in the same plane,

$$\left|\overline{\nabla_v E_1} \times \overline{\nabla_v E_2}\right| = \left|(\overline{\nabla_v E_2} \cdot \hat{e}_1)\overline{\nabla_v E_1} - (\overline{\nabla_v E_1} \cdot \hat{e}_1)\overline{\nabla_v E_2}\right|$$

and

$$\left|\overline{\nabla_v E_2} \cdot \hat{e}_1\right| = \left|b\overline{\nabla_v E_1} \cdot \hat{e}_1\right| = \frac{\left|(b\overline{\nabla_v E_1}) \times (\overline{\nabla_v E_2})\right|}{\left|b\overline{\nabla_v E_1} - \overline{\nabla_v E_2}\right|}$$

The equation for dE/dt with b as an independent parameter is then given by

$$\frac{dE}{dt} = \pm (1 + b^2)^{1/2} \frac{\left|\overline{\nabla_v E_1} \times \overline{\nabla_v E_2}\right|}{\left|b\overline{\nabla_v E_1} - \overline{\nabla_v E_2}\right|} \frac{\left[\overline{F(t)} \cdot \hat{e}_1\right]}{m(t)} \tag{14}$$

For this case in which the thrust vector is confined to a plane normal to the plane of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$ and $E_{min} = 0$, equation (13) or (14) gives dE/dt with either $\hat{e}_1$ or b considered as the independent parameter. The relation is

$$b = \frac{\overline{\nabla_v E_2} \cdot \hat{e}_1}{\overline{\nabla_v E_1} \cdot \hat{e}_1} = \frac{E_2(t)}{E_1(t)}$$

In equations (13) and (14) only the quantity $\frac{\overline{F}(t)}{m(t)} \cdot \hat{e}_1$ varies with time. The similarity of these equations to that for the one-dimensional error (eq. (4)) is apparent.

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Expansion of $\overline{\nabla_v E}$ in Various Coordinate Systems

The vector $\overline{\nabla_v E}$ is the gradient of the scalar function E and as such is independent of the coordinate system chosen. For the examples in this report, the coordinate system shown in figure 3 was used. The unit vector φ is normal to r in the direction of increasing φ. The velocity makes an angle γ with the direction $\hat{\varphi}$. The unit vector n_1 is normal to $\hat{v}$ in the direction of increasing γ. If $E = E(r,\varphi,v,\gamma)$, then

$$dE = E_r\, dr + E_\varphi\, d\varphi + E_v\, dv + E_\gamma\, d\gamma$$

$$= \overline{\nabla_r E} \cdot \overline{dr} + \overline{\nabla_v E} \cdot \overline{dv}$$

Since $\overline{dr} = dr\, \hat{r} + r\, d\varphi\, \hat{\varphi}$ and $\overline{dv} = dv\, \hat{v} + v(d\gamma - d\varphi)\, \hat{n}_1$,

$$E_r\, dr + E_\varphi\, d\varphi + E_v\, dv + E_\gamma\, d\gamma = \overline{\nabla_r E} \cdot (dr\, \hat{r} + r\, d\varphi\, \hat{\varphi}) +$$

$$\overline{\nabla_v E} \cdot [dv\ v + v(d\gamma - d\varphi)\, \hat{n}_1]$$

[redacted]

Equating coefficients of the independent differentials dr, $d\varphi$, dv, and $d\gamma$ yields

$$E_r = \overline{\nabla_r E} \cdot \hat{r}$$

$$E_\varphi = \overline{\nabla_r E} \cdot r\hat{\varphi} - \overline{\nabla_v E} \cdot v\hat{n}_1$$

$$E_v = \overline{\nabla_v E} \cdot \hat{v}$$

$$E_\gamma = \overline{\nabla_v E} \cdot v\hat{n}_1$$

Thus,

$$\left.\begin{aligned} \overline{\nabla_v E} &= E_v\ \hat{v} + (1/v)\ E_\gamma\ \hat{n}_1 \\ \overline{\nabla_r E} &= E_r\ \hat{r} + (1/r)(E_\gamma + E_\varphi)\ \hat{\varphi} \end{aligned}\right\} \qquad (15)$$

For the two-dimensional errors, the examples presented in this report used the errors in latitude and longitude for E_1 and E_2. Each error is expressed functionally as $E = E(r,\varphi,\theta,v,\gamma,\alpha)$, where φ and θ are the longitude and colatitude, respectively, γ is the angle between velocity and the horizontal plane, and α is the azimuthal direction measured counterclockwise from the south. By a method similar to that used previously for motion in a plane,

$$\overline{\nabla_v E} = E_v\ \hat{v} + \frac{1}{v}\ E_\gamma\ \hat{n}_1 - \frac{1}{v\cos\gamma}\ E_\alpha\ \hat{n}_2 \qquad (16)$$

The unit vector $\hat{n}_1$ is normal to $\hat{v}$ in the plane of $(\hat{v},\hat{r})$ and in the direction of increasing γ. The unit vector $\hat{n}_2$ completes the right-hand orthogonal triad $(\hat{v},\hat{n}_1,\hat{n}_2)$.

DISCUSSION

Comparison of Thrust Cutoff and Thrust Vector Rotation

The relative positions of the vectors $\overline{v}(t)$, $\overline{F}(t)$, and $\overline{\nabla_v E}(t)$ for motion in a plane are shown in figure 4, near the burnout point for a typical ICBM trajectory. The angle between $\overline{v}(t)$ and $\overline{F}(t)$ near burnout is probably very small. If the angle between $\overline{F}(t)$ and $\overline{\nabla_v E}(t)$ is $\beta(t)$, equation (4) can be written as

$$\left[\frac{d}{dt}\,E(t)\right]_{\text{along trajectory}} = \left|\nabla_v E(t)\right| \frac{F(t)}{m(t)} \cos\beta(t) \qquad (17)$$

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In this form it is evident that reducing $F(t)$ to zero and reducing $\cos \beta(t)$ to zero (i.e., $\beta(t) \rightarrow 90^{\circ}$) have corresponding effects on the miss distance.

There is thus a choice of two devices by which to fix the miss distance. The use of thrust vector rotation has certain possible advantages and disadvantages. Among the possible advantages are:

(1) "Cutoff" is reversible.

(2) Thrust vector rotation to an angle near $\beta = 90^{\circ}$ can be used for vernier control; that is, $\beta < 90^{\circ}$ will increase E, $\beta > 90^{\circ}$ will decrease E.

(3) Excess thrust can be used, instead of wasted; for example, to increase the loft angle of the trajectory without changing the miss distance. An equation exactly similar to that given for miss distance can be written for any other impact parameter, such as reentry angle, in order to calculate this effect.

(4) For certain propulsion systems, such as solid and nuclear rockets, cutoff of thrust may be very difficult.

(5) The information required by the control system for proper thrust vector rotation is the same as that required for thrust cutoff. The components of the vector $\overline{\nabla_v E}$ occur in the general linearized expansion for dE in terms of $d\overline{v}$, $d\overline{r}$, and dt. The missile attitude is not absolutely required for thrust cutoff but is probably available for purposes of approximate trajectory control.

A possible disadvantage in the use of thrust vector rotation, as shown later, is that if it were to be used in the same way as thrust cutoff is now used, the required missile rotational speed would have to be of the same order as the reciprocal of the thrust cutoff time. This would require very high rotational speeds. This suggests that more anticipatory action would be required for thrust vector rotation. There would then be some loss in the maximum range of the missile.

Equation (17) is useful to calculate the miss distance accrued during the cutoff procedure because, as is shown later, the vector $\overline{\nabla_v E}(t)$ is relatively constant around the expected burnout condition. Thus, considering thrust cutoff, with an exponential thrust decay after time t_0 with time constant τ

$$F(t) = F(t_0)e^{-(t-t_0)/\tau}$$

[redacted]

and the integral of equation (17) gives

$$E(\text{impact}) - E(t_0) = |\nabla_v E(t_0)| \cos \beta(t_0) \frac{F(t_0)}{m(t_0)} \tau \qquad (18)$$

Considering cutoff by thrust vector rotation, beginning at t_0 at constant $\dot{\beta}$ to $\beta = 90^{\circ}$, gives

$$E(\text{impact}) - E(t_0) = |\nabla_v E(t_0)| \left[1 - \sin \beta(t_0)\right] \frac{F(t_0)}{m(t_0)} \frac{1}{\dot{\beta}} \qquad (19)$$

A comparison of equations (18) and (19) shows that the required rotational speed $\dot{\beta}$ must be of the order of $1/\tau$ if thrust vector rotation is to be used in the same way as thrust cutoff is now used. This would indicate extremely high rotational rates, which suggest that more anticipatory action is required for thrust vector rotation. There would then be some loss in maximum range due to this anticipatory action. If $\dot{\beta}$ is much smaller than $1/\tau$, rotation must start at a time approximately equal to $\frac{(\pi/2) - \beta_0}{\dot{\beta}}$ early, and the loss in maximum range is approximately

$$|\nabla_v E(t_0)| \frac{F(t_0)}{m(t_0)} \frac{1}{\dot{\beta}} \left(\pi/2 \cos \beta_0 - 1 + \sin \beta_0 - \beta_0 \cos \beta_0\right) \qquad (20)$$

Some results of calculations from equations (18), (19), and (20), using $|\nabla_v E|$ values presented later for a range of 5400 nautical miles and for $\beta_0 = 0^{\circ}$, $F_0/m_0 = 10g$, are presented in table I. For thrust cutoff with a change in miss distance during cutoff of 5 nautical miles, cutoff time constants of from 15 to 19 milliseconds are required. For thrust vector rotation to be used in the same manner as cutoff, extremely high rotational rates of from 52.6 to 66.7 radians per second would be required. For more moderate rotational rates, anticipation is required and the corresponding maximum loss in maximum range is shown in table I. It is believed that average rotational rates greater than 2 radians per second are feasible, which would mean less than a 100-nautical-mile loss in maximum range.

For the two-dimensional errors on the rotating earth the functions E_1 and E_2 are considered to be the errors in latitude and longitude, respectively. For the thrust vector confined to a plane normal to the plane of $\overline{\nabla_V E_1}$ and $\overline{\nabla_V E_2}$ and orientated so that the possible impact

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points on the earth trace a straight line through the origin on the E_1, E_2 plane, equation (14) applies to the radial error E. If β is the angle between $\overline{F}$ and $\overline{e}_1$,

$$\frac{dE}{dt} = \pm (1 + b^2)^{1/2} \frac{|\overline{\nabla_v E_1} \times \overline{\nabla_v E_2}|}{|b\overline{\nabla_v E_1} - \overline{\nabla_v E_2}|} \frac{F(t)}{m(t)} \cos \beta(t) \tag{21}$$

The parameter b represents the arbitrary initial condition

$$b = \frac{E_2(t)}{E_1(t)} = \frac{E_2(0)}{E_1(0)}$$

For thrust vector rotation at constant $\dot{\beta}$, the error accumulated during rotation is

$$E_{(impact)} - E(t_0) = (\pm)(1 + b^2)^{1/2} \frac{|\overline{\nabla_v E_1} \times \overline{\nabla_v E_2}|}{|b\overline{\nabla_v E_1} - \overline{\nabla_v E_2}|} \frac{F(t_0)}{m(t_0)} \frac{1}{\dot{\beta}} (1 - \sin \beta_0) \tag{22}$$

For moderate $\dot{\beta}$, anticipation is required and rotation must start at a time approximately equal to $\frac{(\pi/2) - \beta_0}{\dot{\beta}}$ early, and the loss in maximum range is approximately

$$(1 + b^2)^{1/2} \frac{|\overline{\nabla_v E_1} \times \overline{\nabla_v E_2}|}{|b\overline{\nabla_v E_1} - \overline{\nabla_v E_2}|} \frac{F(t_0)}{m(t_0)} \frac{1}{\dot{\beta}} \left(\frac{\pi}{2} \cos \beta_0 - 1 + \sin \beta_0 - \beta_0 \cos \beta_0\right) \tag{23}$$

Some results of calculations from equations (22) and (23) using values of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$ for a range of 5150 nautical miles, $\beta_0 = 0^\circ$, and $F_0/m_0 = 10g$ are presented in table II. If the initial conditions are such that there is no cross-range error ($b = \infty$) the results are the same as for motion in a plane (table I). If there is no initial down-range error ($b = 0$), the requirements on rotational speeds are reduced by a factor of 6.6.

Calculations of $\overline{\nabla_v E}$

The components of the vector $\overline{\nabla_v E}$ were calculated for both planar motion on a nonrotating earth and for the two-dimensional errors on the

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rotating earth. As the calculations were made merely to study the feasibility of thrust vector rotation, idealized conditions were assumed where the oblateness of the earth and reentry drag were ignored.

For planar motion the components of $\overline{\nabla_v E}$ are formed from the partial derivatives of the range equation with respect to burnout conditions for various burnout conditions. For the two-dimensional errors on the rotating earth, IBM 704 digital computer solutions were obtained for the basic trajectory equations and the corresponding adjoint equations. The end conditions in the solution of the adjoint equations directly give the partial derivatives that are used in forming the components of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$. The conservation of energy, the conservation of angular momentum, and the first integral of the adjoint equations were used as checks on the numerical solutions.

For planar motion, figures 5 and 6 give $|\overline{\nabla_v E}|$ and $\partial E/\partial v$ plotted against burnout velocity with lines of constant γ and lines of constant range. A burnout altitude of 110 nautical miles is used. The angle between the velocity vector and $\overline{\nabla_v E}$ is

$$\cos|\overline{\nabla_v E},\overline{v}| = \frac{\partial E/\partial v}{|\overline{\nabla_v E}|}$$

If the thrust direction is known relative to the velocity, the angle β between the thrust vector and $\overline{\nabla_v E}$ can then be found.

An inspection of figure 5 at a γ of 30° and a range of 5400 nautical miles reveals that for a change in velocity at a constant γ of 300 feet per second, corresponding to an acceleration of 10g's for about 1 second, there is a 3 percent change in $|\overline{\nabla_v E}|$. From this it appears that the assumption of a relatively constant $|\overline{\nabla_v E}|$ during "cutoff" is valid.

It was seen that the required $\dot{\beta}$ or $1/\tau$ is proportional to $|\overline{\nabla_v E}|$. Figure 5 reveals that although "cutoff" must be faster for the longer ranges there is only a factor of about 4 in comparing 5400- and 1800-nautical-mile ranges at points of minimum burnout velocity. For the 5400-nautical-mile-range line, the lofted, higher velocity trajectory will help alleviate cutoff. For the 1800-nautical-mile range, minimum $|\overline{\nabla_v E}|$ and minimum burnout velocity almost coincide.

For the two-dimensional errors on the rotating earth, figures 7 and 8 give $|\overline{\nabla_v E_\theta}|$ and $|\overline{\nabla_v E_\varphi}|$ plotted against range for lines of constant

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impact velocity (neglecting reentry effects). The case shown is for a burnout altitude of 110 nautical miles and an impact velocity for the basic trajectory due east ($\alpha = 90^{\circ}$). The solid lines indicate impact at the equator, and the dashed lines indicate impact at a latitude of 45°. For these cases, the angle between the two vectors $\overline{\nabla_V E_\theta}$ and $\overline{\nabla_V E_\varphi}$ is extremely close to 90°.

The cross-range sensitivity is not very dependent on range, giving values of $|\overline{\nabla_V E_\theta}|$ from 750 to 1000 seconds for ranges from 1200 to 5000 nautical miles. The down-range sensitivity is similar to that for planar motion. For low ranges (e.g., 1200 nautical miles) the two sensitivities are about equal. For large ranges (e.g., 5000 nautical miles) the down-range sensitivity is greater by a factor of about 6, as previously noted.

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SUMMARY OF RESULTS

In a study of an alternate method to thrust termination for ballistic missiles whereby the thrust is rotated to the insensitive direction the following results were obtained:

1. The feasibility of using thrust vector rotation to the insensitive direction for cutoff of ballistic missiles is shown.

2. Among the chief advantages of this method are:

   (a) Elaborate thrust termination devices are avoided.

   (b) "Cutoff" is reversible.

   (c) Thrust vector rotation near the insensitive direction can be used for vernier control.

   (d) Excess thrust might be used to improve some other impact or trajectory parameter.

   (e) The information required for proper thrust rotation is the same as that required for thrust cutoff.

3. The chief problem of this method would be that for moderate missile rotational speeds there would be some loss in maximum range.

4. A relatively simple equation is derived expressing the effect of thrust vector rotation on the miss distances. For the two-dimensional errors on the rotating earth there is a unique direction of thrust, normal to two defined vectors, which fixes the impact point on the earth.

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If during thrust vector rotation the thrust vector is confined to a plane that is normal to the plane of these two defined vectors, then the possible impact points trace a straight-line path on the earth.

5. The requirements on thrust cutoff times and missile rotational speeds were compared. The extremely high rotational speeds required indicate that anticipatory action would be used. The resulting maximum loss in maximum range, for a 5400-nautical-mile missile, would be less than 100 nautical miles if missile rotational speeds greater than 2 radians per second were available.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, August 21, 1958

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TABLE I. - RESULTS OF CALCULATIONS FROM EQUATIONS (18), (19), AND (20)

[Range, 5400 nautical miles; $\beta_0 = 0$; thrust-mass ratio, $F_0/m_0 = 10g$.]

Thrust cutoff		
$E_{impact} - E_0$, nautical mile	Path angle, deg	Required time constant, τ, sec
5	22	0.015
5	30	.019
Thrust vector rotation		
$E_{impact} - E_0$, nautical mile	Path angle, deg	Required rotational rate, $\dot{\beta}$, radian/sec
5	22	66.7
5	30	52.6
Maximum loss in maximum range, nautical mile	Path angle, deg	Required rotational rate, $\dot{\beta}$, radian/sec
25	22	7.6
50	22	3.8
100	22	1.9

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TABLE II. - RESULTS OF CALCULATIONS

FROM EQUATIONS (22) AND (23)

[Range, 5150 nautical miles; $\beta_0 = 0$; thrust-mass ratio, $F_0/m_0 = 10g$ at impact; $\theta = 45^\circ$, $\gamma = 20^\circ$, $\alpha = 90^\circ$.]

$E_{impact} - E_0$, nautical mile	Slope, b	Required rotational rate, $\dot{\beta}$, radian/sec
5	0	10.13
5	1	14.16
5	∞	66.9
Maximum loss in maximum range, nautical mile	**Slope, b**	**Required rotational rate, $\dot{\beta}$, radian/sec**
25	∞	7.6
50	∞	3.8
100	∞	1.9

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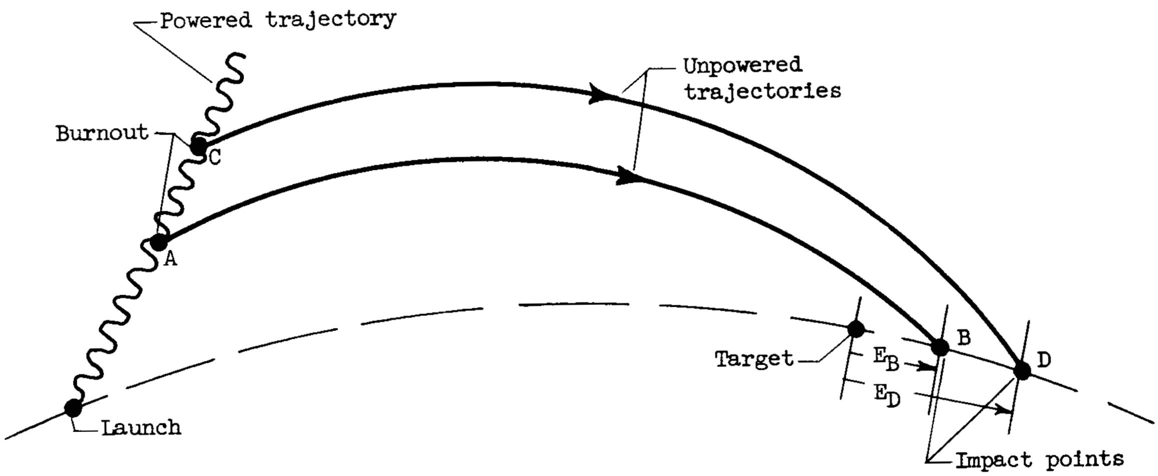

Figure 1. - Schematic drawing of general features of trajectories.

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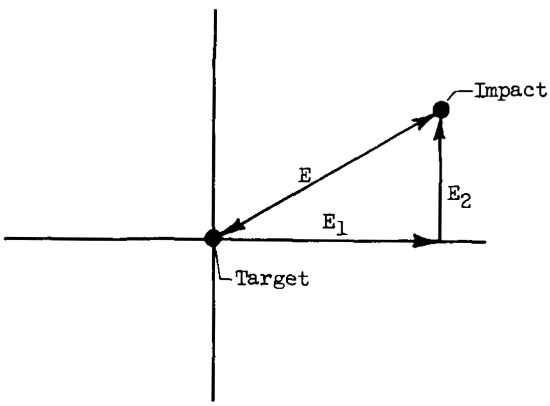
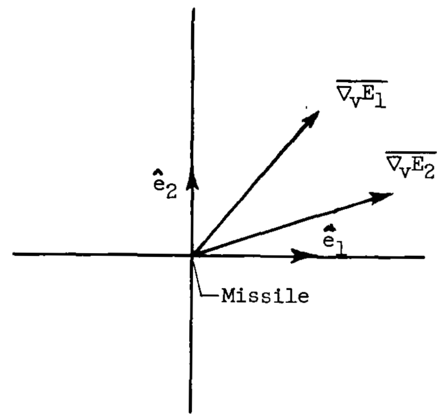
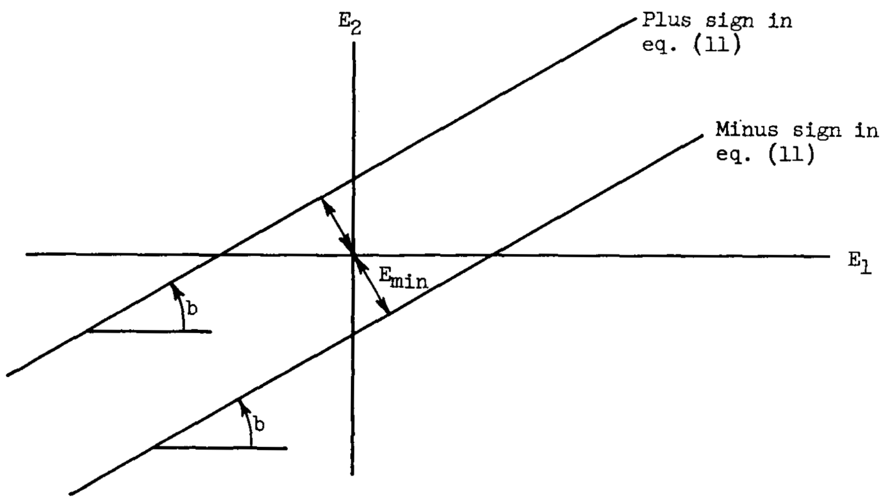


(a) Errors in coordinate system relative to target fixed on the earth.



(b) Plane of $\overline{\nabla_v E_1}$ and $\overline{\nabla_v E_2}$ at missile position.



(c) Path of possible miss distances when thrust vector is confined to plane normal to plane of $\overline{\nabla_v E_1}$, $\overline{\nabla_v E_2}$.

Figure 2. - Two-dimensional miss distances and vector sensitivities.

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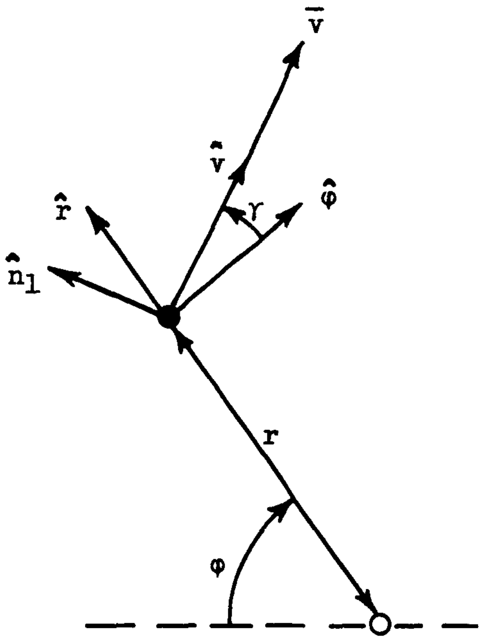

Figure 3. - Coordinate system used in examples of planar motion.

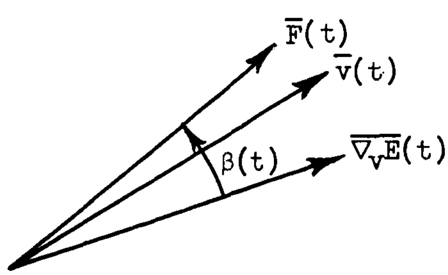

Figure 4. - Relative direction of vector sensitivity $\overline{\nabla_v E}$ near burnout for typical case.

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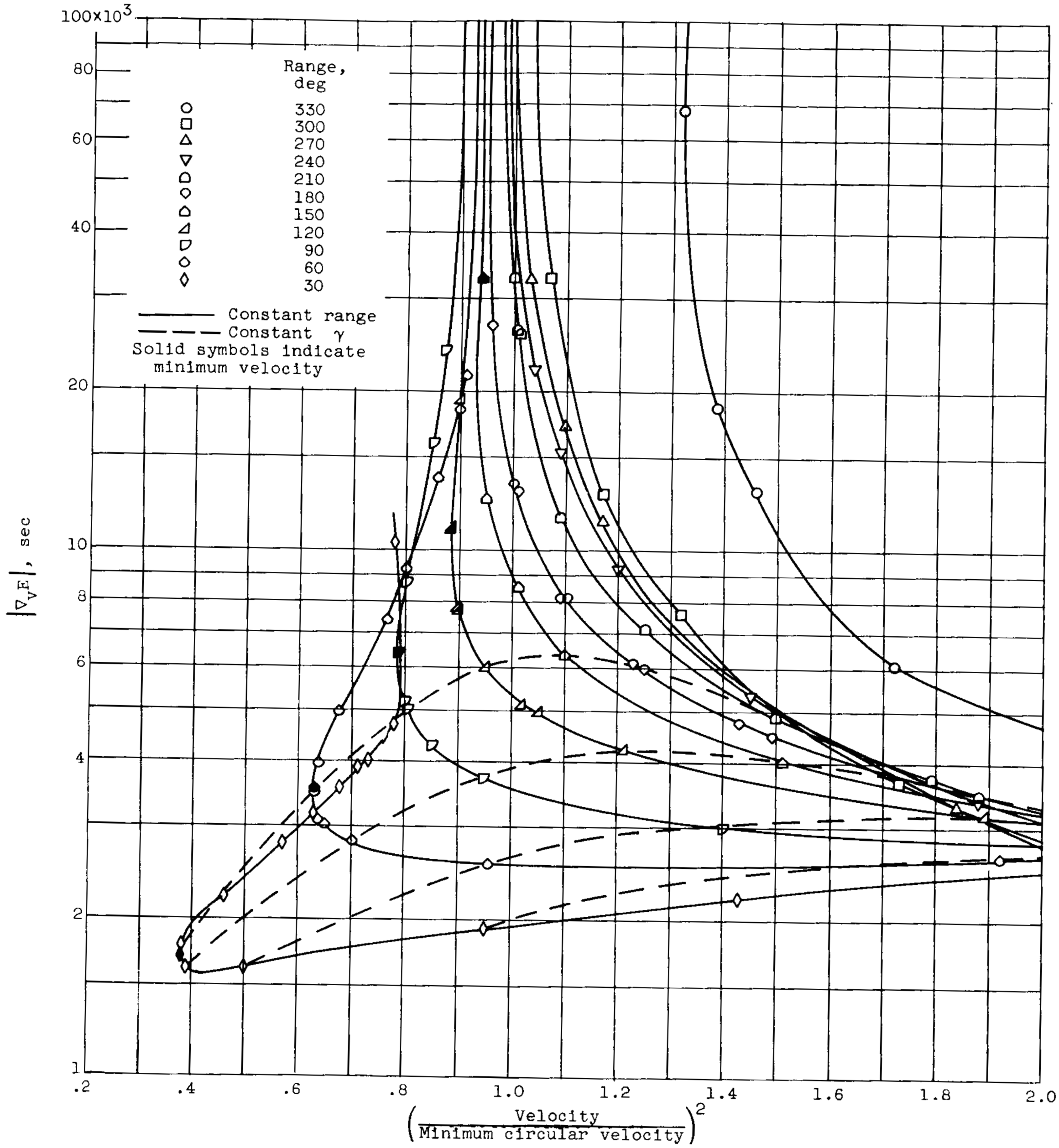

Figure 5. - Absolute value of vector sensitivity. Planar motion; nonrotating earth; burnout altitude, 110 nautical miles.

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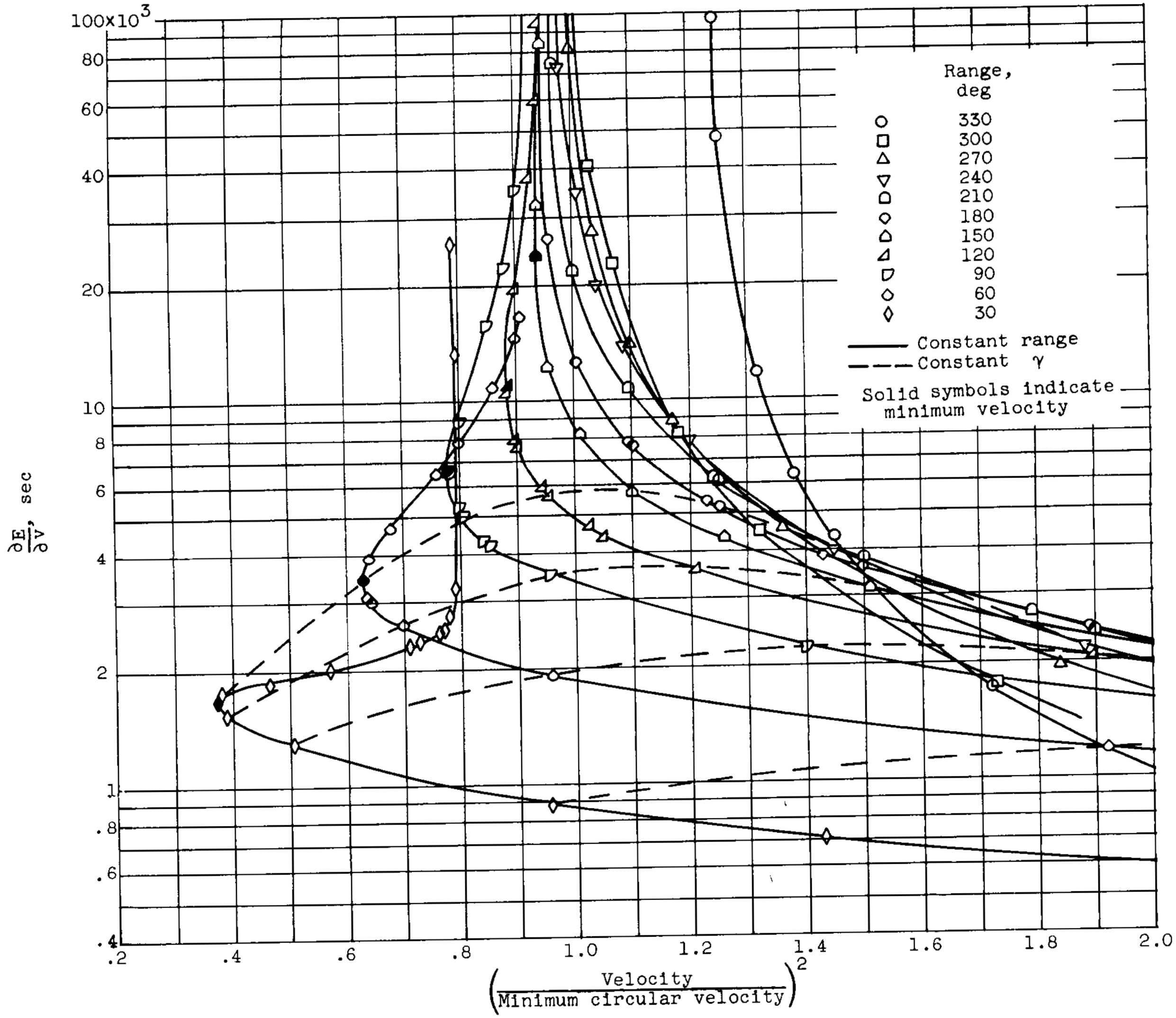

Figure 6. - Component of vector sensitivity $\overline{\nabla_V E}$ in direction of velocity. Planar motion, nonrotating earth; burnout altitude, 110 nautical miles.

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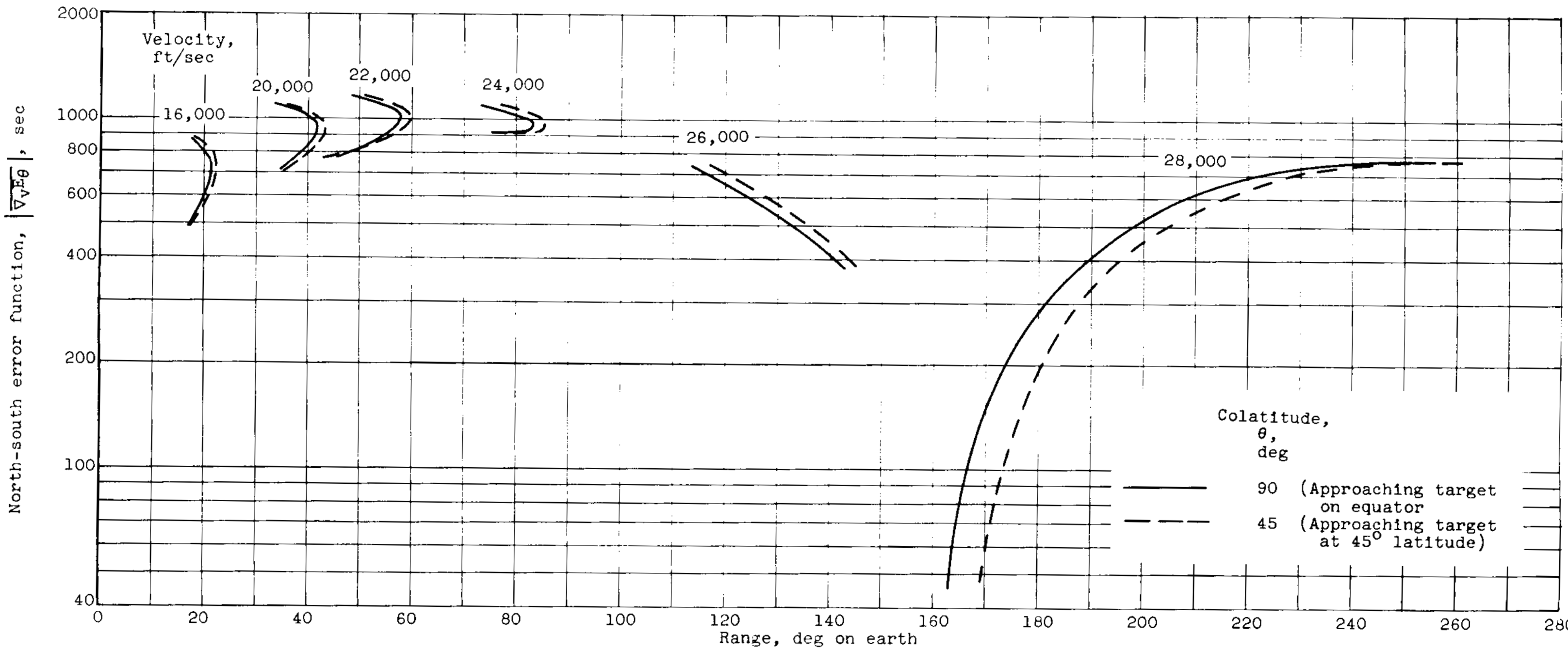

Figure 7. - Absolute value of vector sensitivity $\overline{\nabla_V E_\theta}$ for cross-range error. Approaching target from west; rotating earth; azimuth, 90°; burnout altitude, 110 nautical miles.

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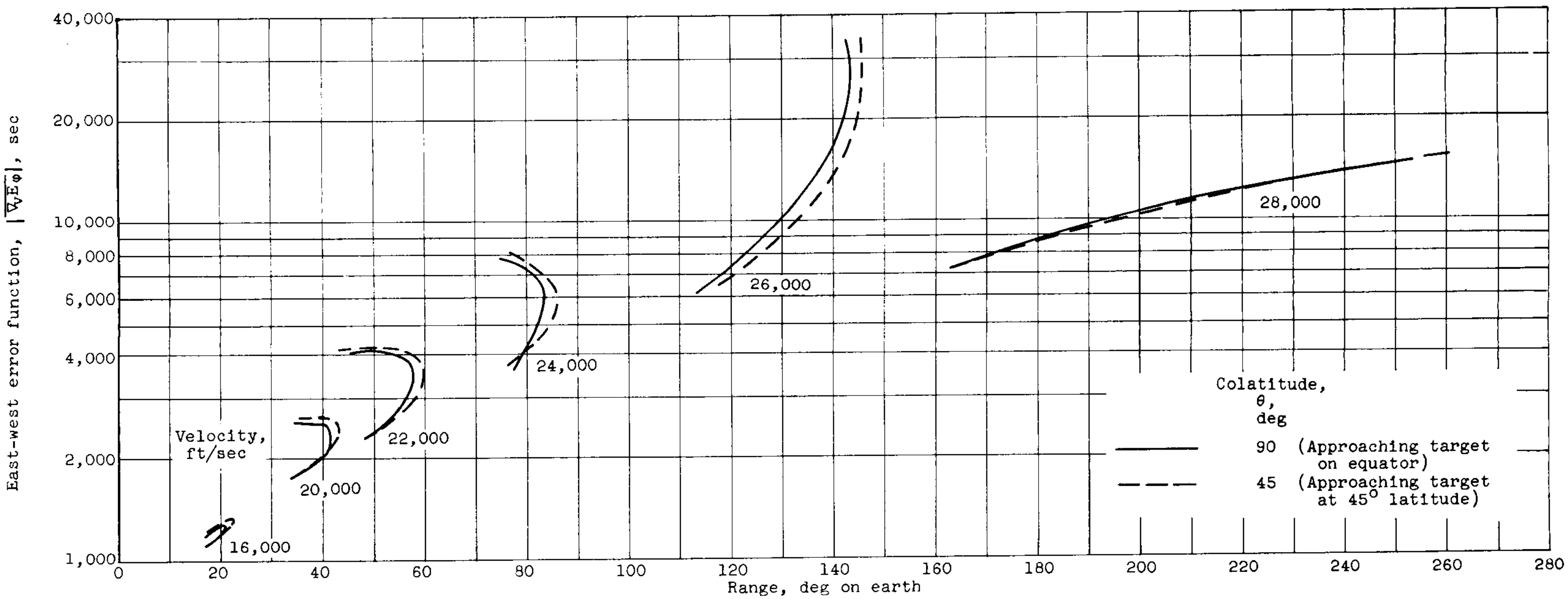

Figure 8. - Absolute value of vector sensitivity $\overline{\nabla_V E_\varphi}$ for down-range error. Approach target from west; rotating earth; azimuth, 90°; burnout altitude, 110 nautical miles.

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